ESTIMATING THE ERROR COMMITTED IN PASSING FROM THE EXACT EQUATIONS OF THE THEORY OF ELASTICITY TO THE EQUATIONS OF THE PLANE STATE OF STRESS

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For thin bodies (shells and plates) the three-dimensional equations of the theory of elasticity are reduced to approximate two-dimensional ones. The error committed in this procedure is usually estimated by comparing a given approximation with the following one. The estimates obtained in this way are expressed in terms of a certain power of the relative thickness.

Such estimates for a plate were obtained by the symbolic method in paper [1] and by the asymptotic method in paper [2].

The investigations carried out in [3 and 4], based on the first method, have indicated that for certain boundary value problems there is no improvement of accuracy as predicted by these estimates in passing from the Kirchhoff-Love approximation to the next following approximations, whereby the contribution of the following approximations may be larger than from the preceding ones, i.e. this method may lead to a decrease of the error. In the present study, with the example of the equations of the state of plane stress, it is shown what the reason for this decrease may be. For these equations the estimate is effected by means of comparing the exact solutions of a known class [5], which are obtained most simply by expanding the displacement vector into Legendre polynomials [6]. The application of such an expansion to the deduction of approximate equations turns out to be useful for several purposes [4 and 7].

The characteristic feature of the result obtained consists in that the estimate is multiplied by the logarithm of the relative thickness which strongly decreases the accuracy. The possibility is not excluded that such a factor, which grows with the decrease of thickness, may occur in other cases also, but it does not appear in the estimates obtained by comparing a given and the following approximation because in these approximations the form of the dependence on relative thickness is prescribed a priori.

1. Fundamental relations. We consider a plane plate of thickness $2h$, bounded by a cylindrical surface, normal to the plate. The plane of Cartesian coordinates x_1, x_2 will be placed into the middle plane of the plate and the dimensionless coordinates $\xi_i = x_i / h$ $(i = 1, 2, 3)$ are introduced.

In coordinates ξ_i the edge boundary will be designated by Γ, the line of its intersection in the plane ξ_1, ξ_2 by γ, and the tree-dimensional region, occupied by the plate, by Ω. Assuming that the displacements are small, as compared to the thickness of the plate, we write down the equations of the theory of elasticity in the form

$$(\lambda+\mu)\frac{\partial\theta}{\partial\xi_i}+\mu\Delta u_i=0, \qquad \theta=\frac{\partial u_i}{\partial\xi_i} \tag{1.1}$$

Here λ and μ are Lamé coefficients, $\{u_i\}$ is the displacement vector, and, as always, the summation convention over dummy indices is applicable. It is not difficult to see that if the displacement vector $\{u_i\}$ is represented by the series

$$u_i=\sum_{r=0}^{2} u_{ir}(\xi_1,\xi_2)\,P_r(\xi_3) \tag{1.2}$$

where $P_r(z)$ is the rth Legendre polynomial ($P_0=1$, $P_1=z$, $P_2=(3z^2-1)/2$), then it will satisfy Equations (1.1) exactly if

$$(\lambda^*+\mu)\frac{\partial\theta_0}{\partial\xi_\alpha}+\mu\Delta_2 u_{\alpha 0}=0, \qquad \theta_0=\frac{\partial u_{\alpha 0}}{\partial\xi_\alpha}, \qquad \lambda^*=\frac{2\lambda\mu}{\lambda+2\mu} \tag{1.3}$$

$$u_{\alpha 2}=\frac{\lambda}{3(\lambda+2\mu)}\frac{\partial\theta_0}{\partial\xi_\alpha}, \qquad u_{31}=-\frac{\lambda}{\lambda+2\mu}\,\theta_0, \qquad u_{30}=u_{32}=u_{\alpha 1}=0 \tag{1.4}$$

Here and in the sequel the tensor indices, indicated by Greek letters, take on the values 1 and 2, and Δ_2 is the two-dimensional Laplace operator. Further, the displacement vector determined by Equations (1.3) and (1.4) satisfies the condition of vanishing boundary traction on the planes $\xi_3=\pm 1$.

Equations (1.3), which have to be satisfied by the principal terms of the expansion (1.2) of the displacement vector, represent the equations of the plane state of stress. Using their solution, we may evaluate by Formulas (1.4) the quantities u_{31} and $u_{\alpha 2}$ and, as will be shown, the error may be estimated by these quantities.

For comparison it is natural to select an exact solution of the three-dimensional equations of the theory of elasticity u_i^* with conditions on Γ, not depending on ξ_3, and for vanishing either transverse component of displacement u_3^*, or transverse component of loading; thereby the loading is absent on the planes $\xi_3=\pm 1$.

For definiteness we assume that on Γ the displacements are prescribed

$$u_\alpha^*|_\Gamma=\varphi_\alpha(\xi_1,\xi_2), \qquad u_3^*|_\Gamma=0 \tag{1.5}$$

The cases for other boundary conditions are investigated analogously.

2. Estimating the error. (1). The vector u_i, approximate for problem (1.1),(1.5), is determined by the series (1.2) and the relations (1.3) and (1.4), and will be found from condition

$$u_{\alpha 0}|_\gamma=\dot{\varphi}_\alpha(\xi_1,\xi_2) \tag{2.1}$$

which, obviously, determines the vector uniquely.

Let us form the difference $\delta u_i=u_i-u_i^*$. The difference satisfies the three-dimensional equations of the theory of elasticity (1.1), the condition of vanishing loadings on the planes $\xi_3=\pm 1$ and the following conditions on Γ

$$\delta u_\alpha|_\Gamma=u_{\alpha 2}|_\gamma P_2(\xi_3)=\frac{\lambda}{3(\lambda+2\mu)}\left(\frac{\partial\theta_0}{\partial\xi_\alpha}\right)_\gamma P_2(\xi_3) \tag{2.2}$$

$$\delta u_3|_\Gamma=u_{31}|_\gamma P_1(\xi_3)=-\frac{\lambda}{\lambda+2\mu}\,\theta_0|_\gamma P_1(\xi_3)$$

2) Let us consider first the bounded plate without openings. Its longitudinal dimension L shall be assumed in all directions to be of the same order and very much larger than the thickness of the plate $2h$, i.e. $(L/h)=\Lambda\gg 1$. We extend the plate beyond Γ to infinity and conserve the displacements δu_i on Γ, determined by conditions (2.2). These displacements will be smoothly

continued beyond Γ such that they vanish at a distance of order 1 from Γ and satisfy the vanishing of the loading of the planes $\xi_3 = \pm 1$.

The solution for δu_i in Ω will remain thereby the same, but over the whole infinite plate the vector δu_i will satisfy the nonhomogeneous equations

$$(\lambda + \mu) \frac{\partial \delta\theta}{\partial \xi_i} + \mu \Delta \delta u_i = f_i, \qquad \delta\theta = \frac{\partial \delta u_i}{\partial \xi_i} \tag{2.3}$$

in which the body forces f_i are different from zero only in a narrow zone D adjoining Γ, whose transverse dimension is of the order of 1. From the condition that the displacements δu_i approach zero smoothly at a distance of order 1 beyond Γ and from the form of their dependence on ξ_3 in Γ, it follows that after differentiating them twice with respect to ξ_3 their order is not changed. Therefore, considering (2.2) and (2.3) we obtain

$$f_i = O\left\{\max\left[\theta_0, \frac{\partial\theta_0}{\partial\xi_\alpha}, \frac{\partial^2\theta_0}{\partial\xi_\alpha\partial\xi_\beta}, \frac{\partial^3\theta_0}{\partial\xi_\alpha\partial\xi_\beta\partial\xi_\gamma}\right]_\gamma\right\} \tag{2.4}$$

for the order f_i .

Using Green's tensor for the infinite plate, we obtain from (2.3)

$$\delta u_i (P) = \int_D G_{ik} (P, Q) f_k (Q) \, d\Omega_Q \tag{2.5}$$

Here P and Q are the points of the plate, whereby P is arbitrary and Q is within the narrow zone D , in which the forces f_k are different from zero.

The components of Green's tensor $G_{ik}(P,Q)$(see, for example, [5]) for fixed k form the displacement vector at the point P , which satisfies the condition of vanishing loading on the planes $\xi_3 = \pm 1$ and the equilibrium equations for unit body force applied at the point Q and directed along the k-axis. Because of the point Q the components G_{ik} possess a sufficiently weak singularity (of the type $1/\rho$, where ρ is the distance from P to Q), therefore G_{ik}^2 are integrable, and the inequality of Cauchy-Buniakowski may be applied to (2.5). It gives

$$|\,\delta u_i (P)\,| \leqslant \left(\int_D \sum_{k=1}^{3} G_{ik}^2 (P, Q) \, d\Omega_Q\right)^{1/2} \left(\int_D \sum_{k=1}^{3} f_k^2 (Q) \, d\Omega_Q\right)^{1/2} \tag{2.6}$$

For the tensor G_{ik} in the case of the infinite plate considered we could supply an explicit expression [8], but there is no necessity for this.

Indeed, because the planes $\xi_3 = \pm 1$ are free of loading, the force applied at the point Q , which generates G_{ik}, is balanced by stresses on an arbitrary surface surroundung the point Q . Let this be a right circular cylinder with its axis passing through Q . The length of its circumference increases in proportion to ρ . Therefore the stresses acting on its surface decrease inversely proportional to ρ .

If $k = 1, 2$, then the force concentrated at the point Q is directed along the plate and is balanced by stresses $\sigma_{\alpha\beta}$. In the expression for these stresses only G_{3k} is differentiated with respect to ξ_3 . Since differentiation with respect to ξ_3 does not change the order, G_{3k} will decrease as ρ^{-1}. The remaining components of displacement in the expressions for $\sigma_{\alpha\beta}$ are differentiated along the plate, and because their length derivatives decrease as ρ^{-1}, it follows that the quantities themselves increase as $\ln\rho$. This is in complete accord with the statement that when the concentrated force is directed along the plate, the elastic field at infinity approaches the plane state, for which the displacements increase logatithmically in the case of concentrated loading, which is a well-known result.

For $k = 3$ the concentrated force is directed transversely to the plate, and one can show analogously that in this case G_{3k} will increase logarithmically, while $G_{\alpha k}$ will decrease (as ρ^{-1}). Therefore for $\rho \to \infty$

$$\sum_{k=1}^{3} G_{ik}^2 (P, Q) = O (\ln^2 \rho) \tag{2.7}$$

If we now take in (2.6) the point $P = P^*$ in the vicinity of the middle of the plate, then, considering that the nondimensional extent of the plate $\Lambda = L/h$ is much larger than unity, we may use the estimate (2.7). Substituting this estimate in (2.6), the estimate (2.4) for f_k and considering that the transverse dimension of the region D is of the order of unity, while its inplane dimension is of the order Λ, we obtain

$$\delta u_i (P^*) = O\left\{(\Lambda \ln \Lambda) \max\left[\theta_0, \frac{\partial\theta_0}{\partial\xi_\alpha}, \frac{\partial^2\theta_0}{\partial\xi_\alpha\,\partial\xi_\beta}, \frac{\partial^3\theta_0}{\partial\xi_\alpha\,\partial\xi_\beta\,\partial\xi_\gamma}\right]_\gamma\right\} \qquad (2.8)$$

This result is conserved also in the case when the loading is prscribed along Γ, or when on one part of Γ the loading is given and on the other the displacements, provided that the boundary conditions do not depend on ξ_3, and at the corresponding parts of Γ either the transverse force or the transverse displacement vanish.

Thus, if the solution of the problem of the plane state of stress is known, then calculating on the contour the value θ_0 and all its derivatives up to the third order, we can estimate the error in the displacements of the middle part of the plate using (2.8). We note that the estimate (2.8), being a general one, may be strongly increased (for example, for the class of exact solutions used), but it cannot be improved because for solutions, corresponding to the case $f_k(Q) = \text{const}\cdot G_{ik}(P, Q)$, it becomes exact when in (2.6) the equality is realized.

3) Let us consider now the infinite plate with cutouts. Extending the plate with the cutouts, and otherwise proceeding analogously as we described above, we arrive to Formula (2.5) and to the inequality (2.6) in which D must be interpreted as "O(1) in the vicinity" for all openings. Evaluating the error δu_i at the points $P = P^{**}$, situated at large distances from the openings $(\Lambda = (L/h) \gg 1)$, we obtain the estimate (2.8), in which Λ in front of the logarithm should be deleted, because now the dimensions of region D do not depend on Λ. Again, this estimate cannot be improved. It follows that the error in displacements, generally speaking, increases logarithmically with distance from the cutouts.

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